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CALCULUS OF OPERATORS: COVARIANT TRANSFORM AND RELATIVE CONVOLUTIONS

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ABSTRACT. This paper outlines a covariant theory of operators defined on groups and homogeneous spaces. A systematic use of groups and their representations allows to obtain results of algebraic and analytical nature. The consideration is systematically illustrated by a representative collection of examples.

CONTENTS

Calculus of operators on groups and homogeneous spaces has a long history [13, 14, 16, 18, 21–23, 28–31, 34, 35, 49, 56] and is enjoying a recently revived

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interest [3, 15, 43, 50, 54, 61]. There are some missing connections between two periods and the purpose of this presentation is to bridge the gap.

1. Calculus of Pseudodifferential Operators

The theory of pseudo-differential operators (PDO) is an important and profound area of analysis with numerous applications [12, 20, 51, 55, 60]. In the simplest one-dimensional case, a PDO $A = a(x, D)$ is defined from its *Weyl symbol* $a(x, \xi)$—a function on $\mathbb{R}^2$—by the identity [17, (2.3)]:

$$[Au](y) = \int_{\mathbb{R}\times\mathbb{R}} a(\tfrac{1}{2}(y + x), \xi)\, e^{2\pi i(y-x)\xi}\, u(x)\, dx\, d\xi. \tag{1.1}$$

The alternative *Kohn–Nirenberg correspondence* between symbols and operators [17, § 2.2] is provided by a similar formula:

$$[A_{KN}u](y) = \int_{\mathbb{R}\times\mathbb{R}} a(y, \xi)\, e^{2\pi i(y-x)\xi}\, u(x)\, dx\, d\xi. \tag{1.2}$$

There is a natural demand to generalise PDO for other settings. It is common to have several competing approaches for this. We briefly outline two of them.

1.1. Pontryagin Duality and the Fourier Transform.

Historically, the theory of PDO grown out of the study of singular integral operators (SIO), which can be viewed either as convolutions on the Euclidean group or Fourier multipliers. In either case, this prompts a consideration of groups and representation theory. For simplicity, we take $G = (\mathbb{R}, +)$—the abelian group of reals with addition. Its Pontryagin dual—the collection of all unimodular characters $\chi_\xi(x) = e^{2\pi i\xi x}$—is again the abelian group $\hat{G}$ isomorphic to $(\mathbb{R}, +)$ [27, § IV.2.1]. The Fourier transform $\mathcal{F}$ maps a function f from the Schwartz space $S(G)$ to $\hat{f} \in S(\hat{G})$ by the formula:

$$\hat{f}(\xi) = \int_G f(x)\, \overline{\chi_\xi(x)}\, dx = \int_{\mathbb{R}} f(x)\, e^{-2\pi i\xi x}\, dx. \tag{1.3}$$

This map is unitary on $L_2(G)$. Pontryagin duality ensures that the second dual $\hat{\hat{G}}$ is canonically isomorphic to G and provides an expression for the inverse Fourier transform:

$$f(x) = \int_{\mathbb{R}} \hat{f}(x)\, e^{2\pi i\xi x}\, dx. \tag{1.4}$$

Then, one can interpret the formula (1.2) as follows:

$$[Au](y) = \int_{\hat{G}} \chi_\xi(y)\, a(y, \xi) \int_G u(x)\, \overline{\chi_\xi(x)}\, dx\, d\xi, \tag{1.5}$$

where the symbol $a(x, \xi)$ is a function on $G \times \hat{G}$. Since Pontryagin duality and the respective Fourier transforms are readily available for a locally-compact abelian group, this viewpoint generates a related theory of PDO on commutative groups, see [50, Part II].

The situation is different for non-commutative groups. The dual object $\hat{G}$ of a non-abelian group G—the collection of all equivalence classes of irreducible unitary representations—is not a group, in general. One can still define the (operator

valued!) Fourier transform by the formula

$$\hat{f}(\xi) = \int_G f(g)\,\xi(g)\,dg, \qquad \text{where } f \in L_1(G, dg), \quad \xi \in \hat{G}$$

and dg is a left-invariant (Haar) measure on G. The inverse Fourier transform is not as simple as in the commutative case. For example, on a compact group it is:

$$[\mathcal{F}^{-1}F](x) = \sum_{[\xi]\in\hat{G}} \dim(\xi)\,\mathrm{Tr}\,(\xi(x)F(\xi)) \,,$$

where Tr denotes the trace of an operator. Thus, for a compact group an analog of PDO with a symbol $a(x,\xi)$ on $G \times \hat{G}$ can be defined by, [50, (10.19)]:

$$Af(x) = \sum_{[\xi]\in\hat{G}} \dim(\xi)\,\mathrm{Tr}\left(\xi(x)\,a(x,\xi)\,\hat{f}(\xi)\right). \tag{1.6}$$

Similar formulae were used in the context of the Heisenberg group [3] and other nilpotent Lie groups [15].

1.2. **Covariant Transform.** A different approach starts from the observation that operators of spatial shifts $f(t) \mapsto f(t-x)$ and operators of multiplications by exponents $f(t) \mapsto e^{2\pi i y t} f(t)$ (i.e. shifts in the frequency space) generate the Schrödinger representation of the non-commutative Heisenberg group $\mathbb{H}$ [17,21]. As C^∞-manifold $\mathbb{H}$ can be identified with $\mathbb{R}^3$ and the group law is:

$$(s,x,y) * (s',x',y') = (s+s'+\tfrac{1}{2}(xy'-x'y), x+x', y+y'). \tag{1.7}$$

The Schrödinger representation $\mathbb{H}$ is

$$\rho(s,x,y)f(t) = e^{\pi i(2s+y(2t-x))}\,f(t-x). \tag{1.8}$$

We can integrate this representation with the Fourier transform $\hat{\sigma}(x,y)$ of a function $\sigma(q,p)$ on $\mathbb{R}^2$ [17, § 2.1]:

$$\begin{aligned}
[\rho(\hat{\sigma})f](t) &= \int_{\mathbb{R}^2} \hat{\sigma}(x,y)\,\rho(0,x,y)f(t)\,dx\,dy\\
&= \int_{\mathbb{R}^2} \hat{\sigma}(x,y)\,e^{\pi i y(2t-x)}\,f(t-x)\,dx\,dy\\
&= \int_{\mathbb{R}^2} \sigma(q, \tfrac{1}{2}(r+t))\,e^{2\pi i q(r-t)}\,f(r)\,dq\,dr. \qquad (1.9)
\end{aligned}$$

Up to different letters, this is exactly PDO (1.1) with the Weyl symbol σ.

It may not be obvious that an introduction of the non-commutative Heisenberg group produces any advantage over commutative Pontryagin duality. Probably, it explains why this direction, rooted in Weyl's original works and spectacularly developed in [21,22], was not widely adopted (see, however, remarkable exceptions [13,14,17,18,49]).

Benefits, which can be challenged within the classical PDOs, become more explicit when we move to a general setup. A transition to a non-commutative underlining group does not become an issue since non-commutativity is already in the scheme. Thus, the construction of non-abelian PDOs is different in a computational sense rather than conceptually. Moreover, historically PDOs appeared as "SIOs varying from point to point" and these roots were preserved in [13,14]. We will recover them in examples with the Dynin group below.

Such a development is very straightforward for nilpotent Lie groups, as was already hinted in [13, 14, 22]. Thus, the concept of relative convolutions [34] was initially developed in the nilpotent setting. However, the approach is also usable for non-compact non-commutative non-exponential (e.g. semisimple) Lie groups as well. The present paper provides a brief illustration to this claim.

2. Groups and Representations

Our construction is based on groups and representation theory. It is connected to the covariant transform [37–39, 44], which consolidates a large collection of results linked to wavelets/coherent states [34, 35].

2.1. **Main Examples.**

For the sake of brevity, we explicate our approach only by the following four examples. However, they do not exhaust all possible applications.

2.1.1. *The Heisenberg Group.* For simplicity, we use only the smallest one-dimensional *Heisenberg group* $\mathbb{H}$ consisting of points $(s, x, y) \in \mathbb{R}^3$ [17, 22]. The group law on $\mathbb{H}$ is given by (1.7). The Heisenberg group is a non-commutative nilpotent Lie group with the centre

$$Z = \{(s, 0, 0) \in \mathbb{H},\ s \in \mathbb{R}\}.$$

The Lie algebra $\mathfrak{h}$ is realised by the following left-(right-)invariant vector fields:

$$S^{l(r)} = \pm\partial_s, \quad X^{l(r)} = \pm\partial_x - \tfrac{1}{2}y\partial_s, \quad Y^{l(r)} = \pm\partial_y + \tfrac{1}{2}x\partial_s. \tag{2.1}$$

They satisfy to the *Heisenberg commutator relations* $[X, Y] = S$ and $[X, S] = [Y, S] = 0$.

2.1.2. *Abstract Heisenberg–Weyl (AHW) Group.* Let G be a locally compact abelian group. Pontryagin duality tells that the collection $\hat{G}$ of all unitary characters of G is a locally compact group as well. For example [27, § IV.2.1], $\hat{\mathbb{R}} = \mathbb{R}$, $\hat{\mathbb{Z}} = \mathbb{T}$, $\hat{\mathbb{T}} = \mathbb{Z}$, where $\mathbb{T}$ is the group of unimodular complex numbers. The group operations on both G and $\hat{G}$ are denoted by $+$ and their units are written as 0.

We form a new group $\tilde{G}$ as the set $\mathbb{T} \times G \times \hat{G}$ with the group law:

$$(z_1, g_1, \chi_1) * (z_2, g_2, \chi_2) = (z_1 z_2 \chi_2(g_1), g_1 + g_2, \chi_1 + \chi_2),$$

where $z_i \in \mathbb{T}$, $g_i \in G$, $\chi_i \in \hat{G}$, $i = 1, 2$. In general, $\tilde{G}$ is a non-commutative locally compact group. The unit is $(1, 0, 0)$ and the inverse of (z, g, χ) is $(\bar{z}\chi(g), -g, -\chi)$. The centre of $\tilde{G}$ consists of elements $(z, 0, 0)$, $z \in \mathbb{T}$. The left- and right-invariant measures coincide with the product of invariant measures of $\mathbb{T}$, G and $\hat{G}$.

For $G = \mathbb{R}$, the group $\tilde{G}$ is the polarised Heisenberg group [17, § 1.2] in the reduced form [17, § 1.3]. Thus, for a general G, we call $\tilde{G}$ the *abstract Heisenberg–Weyl (AHW) group*. Another basic example of the AHW group is $\tilde{\mathbb{T}} = \mathbb{T} \times \mathbb{T} \times \mathbb{Z}$, with the group law:

$$(z_1, w_1, k_1) * (z_2, w_2, k_2) = (z_1 z_2 w_1^{k_2}, w_1 w_2, k_1 + k_2),$$

where $z_i, w_i \in \mathbb{T}$ and $k_i \in \mathbb{Z}$, $i = 1, 2$ and the operation on $\mathbb{T}$ written as multiplication of complex numbers. Of course, by the Pontryagin duality $\tilde{\hat{G}}$ is isomorphic to $\tilde{G}$, in particular, $\tilde{\mathbb{T}}$ is isomorphic to $\tilde{\mathbb{Z}}$. Our consideration of $\tilde{G}$ shall be compared with [61].

2.1.3. *The Dynin Group.* Consider a Lie algebra $\mathfrak{d}$ spanned by the basis $\{Z, T, U, V, S, X, Y\}$ defined by the following non-vanishing commutators [13]:

$$[X, Y] = S, \qquad [X, U] = Z, \qquad [Y, V] = Z, \tag{2.2}$$

$$[S, T] = Z, \qquad [X, T] = -\tfrac{1}{2}V, \qquad [Y, T] = \tfrac{1}{2}U. \tag{2.3}$$

Thus, the Lie algebra $\mathfrak{d}$ is nilpotent step 3. It is generated by its elements X, Y and T and their commutators. The multiplication on a group $\mathbb{D}$, obtained from $\mathfrak{d}$ by exponentiation, is:

$$\begin{aligned}(z, t, u, v, s, x, y) * (z', t', u', v', s', x', y')&\\ =\Big(&z + z' + \tfrac{1}{2}(st' - s't) + \tfrac{1}{2}(xu' - x'u) + \tfrac{1}{2}(yv' - y'v)\\ &\quad + \tfrac{1}{24}(yx't - xy't + y'xt' - x'yt'),\\ &t + t',\ u + u' + \tfrac{1}{4}(yt' - y't),\ v + v' - \tfrac{1}{4}(xt' - x't),\\ &s + s' + \tfrac{1}{2}(xy' - x'y), x + x', y + y'\Big).\end{aligned} \tag{2.4}$$

A representation dR of $\mathfrak{d}$ appears if we extend the representation of the Lie algebra $\mathfrak{h}$ by the left-invariant vector fields (2.1) with the representation of the additional operators U, V, T, Z as operators of multiplication:

$$dR^U = xI, \qquad dR^V = yI, \qquad dR^T = sI, \qquad dR^Z = I. \tag{2.5}$$

This representation is connected with the algebra generated by convolutions on the Heisenberg group and operators of multiplications by functions, see Example 4.1(iii) below. The group $\mathbb{D}$ was used in papers [13, 14], thus we call it the *Dynin group*. It is a special (but, probably, the most important) example of meta-Heisenberg group [18]. It is also a subgroup of the group studied in [49].

2.1.4. *The group* $SU(1, 1)$. The group $SU(1, 1)$ [46, § IX.1; 57, § 8.1] consists of 2×2 matrices with complex entries of the form $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ and unit determinant: $|\alpha|^2 - |\beta|^2 = 1$. The multiplication is given by matrix multiplication and is not commutative. The maximal compact subgroup K of diagonal matrices $\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$ is isomorphic to the unit circle. Its presence indicates that the group is not exponential, that is the exponent map from the Lie algebra to the group is not a bijection. This is group is not compact.

The group $SU(1, 1)$ acts by Möbius transformations of the unit disk (Example 2.1(iv) below) and is very important in complex analysis, cf. [8]. Intimate connections of other subgroups of $SU(1, 1)$ with the hypercomplex numbers is described in [39, 40, 42], we do not touch this interesting topic in this paper.

2.2. **Induced Representations.** The general scheme of induced representations is as follows, see also [19, Ch. 6; 25, § 13.2; 26, § V.2; 33, § 3.1; 57, Ch. 5].

Let G be a locally compact group and let H be its subgroup. Let $X = H\backslash G$ be the corresponding right coset space and $s : X \to G$ be a continuous function (section) [25, § 13.2] which is a right inverse to the natural projection $p : G \to H\backslash G$. Then, any $g \in G$ has a unique decomposition of the form $g = h * s(x)$ where $x = p(g) \in X$ and $h \in H$. We define the map $r : G \to H$:

$$r(g) = g * s(x)^{-1}, \qquad \text{where } x = p(g). \tag{2.6}$$

Note, that X is a right homogeneous space with the G-action defined in terms of p and s as follows:

$$g : x \mapsto x \cdot g = p(s(x) * g), \tag{2.7}$$

where $*$ is the multiplication on G.

Example 2.1. (i) For the Heisenberg group $\mathbb{H}$ we can consider the subgroup $Z = \{(s, 0, 0) \mid s \in \mathbb{R}\}$. The corresponding homogeneous space is $Z\backslash\mathbb{H} = \{(0, x, y) \mid (x, y) \in \mathbb{R}^{2n}\}$. Using the maps $p : (s', x', y') \mapsto (x', y')$ and $s : (x', y') \mapsto (0, x', y')$ we calculate the action:

$$(s, x, y) : (x', y') \mapsto (x + x', y + y'). \tag{2.8}$$

There is also a subgroup

$$H_x = \{(s, 0, y) \in \mathbb{H} \mid s, y \in \mathbb{R}\} \tag{2.9}$$

and the respective homogeneous space is parametrised by the real line. Using the maps $p : (s', x', y') \mapsto x'$ and $s : x' \mapsto (0, x', 0)$ we find the action of $\mathbb{H}$ on $H_x\backslash\mathbb{H}$:

$$(s, x, y) : x' \mapsto x + x'. \tag{2.10}$$

(ii) For an AHW group $\tilde{G}$, there are also two commutative subgroups: the centre $Z = \{(z, 0, 0) \mid z \in \mathbb{T}\}$ and

$$H_G = \{(z, 0, \chi) \in \tilde{G} \mid z \in \mathbb{T},\ \chi \in \hat{G}\}. \tag{2.11}$$

The natural maps s and respective actions on the homogeneous spaces are similar to the above particular case of $\mathbb{H} \sim \tilde{\mathbb{R}}$.

(iii) For the Dynin group $\mathbb{D}$, consider the commutative subgroup

$$M = \{(z, t, u, v, 0, 0, 0) \in \mathbb{D} \mid (z, t, u, v) \in \mathbb{R}^4\}. \tag{2.12}$$

The homogeneous space $M\backslash\mathbb{D}$ can be identified with $\mathbb{H}$ through the map $s(s, x, y) = (0, 0, 0, 0, s, x, y)$. The corresponding action is in the essence the group law (1.7) of $\mathbb{H}$:

$$(z, t, u, v, s, x, y) \cdot (s', x', y') = (s + s' + \frac{1}{2}(xy' - x'y), x + x', y + y'). \tag{2.13}$$

(iv) For the group $SU(1, 1)$ and its subgroup K we identify $K\backslash SU(1, 1)$ with the open unit disk. Defining maps:

$$p : \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mapsto \frac{\beta}{\alpha} \quad \text{and} \quad s : z \mapsto \frac{1}{\sqrt{1 - |z|^2}} \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix}, \quad \text{where } |z| < 1 \tag{2.14}$$

we deduce the respective action:

$$z \cdot g = \frac{\bar{\alpha} z + \beta}{\bar{\beta} z + \alpha}, \qquad \text{where } g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

This is a linear-fractional (Möbius) transformation of the unit disk [40, Ch. 10].

For G and H, we respectively denote the right Haar measures dg and dh, the corresponding modular functions are Δ_G and Δ_H. Then, there is a measure dx on $X = H\backslash G$ defined up to a scalar factor by the identities [25, § 9.1(5′-6′)]:

$$dg = \frac{\Delta_G(h)}{\Delta_H(h)}\, dx\, dh, \qquad \text{where } g = hs(x). \tag{2.15}$$

The measure dx transforms under G action (2.7) by, see [25, § 9.1(7′-8′)]:

$$\frac{d(x\cdot g)}{dx} = \frac{\Delta_H(h(x,g))}{\Delta_G(h(x,g))}, \qquad \text{where} \quad s(x)g = h(x,g)s(x\cdot g). \tag{2.16}$$

In many cases, e.g. for all nilpotent Lie groups [26, § 3.3.2], unitary representations are induced by characters—one dimensional linear representations—of its subgroups. Thus we present here the induction from a character only. Let $\chi : H \to \mathbb{C}$ be a unitary character of H. Consider the space of functions on G with the property:

$$F(hg) = \chi(h)F(g), \tag{2.17}$$

the space is obviously invariant under right translations. The restriction of the right regular representation: $R(g) : f(g') \mapsto f(g'g)$ to this space is called an *induced representation* in the sense of Mackey [25, § 13.2; 26, § V.2].

Consider the *lifting* $\mathcal{L}_\chi : C_b(X) \to C_b(G)$ of continuous bounded functions:

$$F(g) = [\mathcal{L}_\chi f](g) = \chi(h)f(p(g)), \qquad f(x) \in C_b(X). \tag{2.18}$$

The function $F(g)$ has the property (2.17). The same expression (2.18) defines a bijection from $L_p(X)$ to certain space $L_p^\chi(G)$, which is invariant under right translations. A right inverse map—the *pulling*—$\mathcal{P} : L_p^\chi(G) \to L_p(X)$ is defined by:

$$f(x) = [\mathcal{P}F](x) = F(s(x)), \qquad F(g) \in L_p^\chi(G). \tag{2.19}$$

The norm on $L_p^\chi(G)$ is introduced in such a way that both the lifting and pulling are isometries.

Since $L_p^\chi(G)$ is invariant under the right shifts, lifting and pulling intetwine the restriction $R|_{L_p^\chi(G)}$ of the right regular representation R with the representation $\rho_\chi(g) = \mathcal{P} \circ R(g) \circ \mathcal{L}_\chi$. It is the second form of the induced representation. Its realisation ρ_χ in a space of complex-valued functions on X, cf. [25, § 13.2(7)–(9)] is:

$$[\rho_\chi(g)f](x) = \chi(r(s(x) * g))\, f(x\cdot g), \tag{2.20}$$

where $g \in G$, $x \in X$, $h \in H$ and $r : G \to H$, $s : X \to G$ are maps defined above; $*$ denotes multiplication on G and $x \cdot g$ denotes the action (2.7) of G on X from the right.

For the case of an unimodular group G and an unimodular subgroup $H \subset G$ (which is automatic for a nilpotent G), the representation (2.20) is unitary in $L_2(X)$. In the case of a non-unimodular (sub)group, we need an additional factor $\left[\frac{\Delta_H(h(x,g))}{\Delta_G(h(x,g))}\right]^{\frac{1}{2}}$ to make ρ_χ unitary, cf. (2.16) and [25, § 13.2(3)].

Example 2.2. (i) For the centre Z of $\mathbb{H}$, the map $r : \mathbb{H} \to Z$ is $r(s,x,y) = (s,0,0)$. The character $\chi_\hbar(s,0,0) = e^{2\pi i\hbar s}$ of Z together with the action (2.8) produces the unitary Fock–Segal–Bargmann (FSB) representation [17, § 1.6; 41]:

$$[\rho^F_\hbar(s,x,y)f](x',y') = e^{\pi i\hbar(2s+x'y-xy')}f(x'+x,y'+y). \tag{2.21}$$

We identify a point (x,y) of the homogeneous space $Z\backslash\mathbb{H}$ with the complex number $z = x+iy$. Then, the representation (2.21) can be stated in the complex form:

$$[\rho^F_\hbar(s(z))f](z') = e^{\pi\hbar(z\bar{z}'-\bar{z}z')/2}f(z+z'). \tag{2.22}$$

For the subgroup H_x (2.9), the map $r(s,x,y) = (s+\frac{1}{2}xy,0,y)$. A character $\chi_\hbar(s,0,y) = e^{2\pi i\hbar s}$ and the action (2.10) produce the Shrödinger representation [17, § 1.3; 41]:

$$[\rho_\hbar(s,x,y)f](x') = e^{\pi i\hbar(2s+2x'y+xy)}f(x'+x). \tag{2.23}$$

Clearly, $\rho_1(-s,-x,-y)$ coincides with the representation (1.8). Furthermore, it is known that they are unitary equivalent to FSB representation (2.21).

(ii) For an AHW group $\tilde{G}$, we proceed in a similar fashion. The character $\nu(z,0,0) = z^k$ of the centre induce the representation on $L(G\times\hat{G})$:

$$[\rho^F_k(z,g,\chi)f](g',\chi') = (z\chi(g'))^k\, f(g+g',\chi+\chi'). \tag{2.24}$$

For the subgroup H_G (2.11), the map $r(z,g,\chi) = (z,0,\chi)$ and the character $\nu(z,0,\chi) = z^k$ induces the representation on $L(G)$:

$$[\rho_k(z,g,\chi)f](g') = (z\chi(g'))^k\, f(g+g'). \tag{2.25}$$

(iii) For the subgroup M (4.8) of $\mathbb{D}$, the map

$$r(z,t,u,v,s,x,y) = (z+\tfrac{1}{2}st+\tfrac{1}{2}xu+\tfrac{1}{2}yv,\ t,\ u+\tfrac{1}{4}yt,\ v-\tfrac{1}{4}xt,\ 0,\ 0,\ 0).$$

The character $\chi(z,t,u,v,0,0,0) = e^{2\pi i hz}$ of M induces the representation:

$$\begin{aligned}[\rho_h(z,t,u,v,s,x,y)f](s',x',y') &= e^{\pi i h(2z+2s't+st+\frac{1}{4}(x'y-xy')t+(2x'+x)u+(2y'+y)v)}\\ &\quad\times f(s+s'+\tfrac{1}{2}(x'y-xy'),x+x',y+y').\end{aligned} \tag{2.26}$$

In particular, the action of the subgroup M reduces to multiplication:

$$[\rho_h(z,t,u,v,0,0,0)f](s',x',y') = e^{2h\pi i(z+ts'+ux'+vy')}f(s',x',y'). \tag{2.27}$$

On the other hand, the operator $\rho_h(0,0,0,0,s,x,y)$ is the shift (2.13) by (s,x,y) on $\mathbb{H}$. The corresponding infinitesimal actions are (2.1) and (2.5). This representation was used in [13], see also [18, § 3].

(iv) For $G = SU(1,1)$ and $H = K$ we calculate $r\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix} = \begin{pmatrix}\frac{\alpha}{|\alpha|} & 0\\ 0 & \frac{\bar{\alpha}}{|\alpha|}\end{pmatrix}$. Let a character χ of K be $\chi\begin{pmatrix}e^{i\phi} & 0\\ 0 & e^{-i\phi}\end{pmatrix} = e^{-2i\phi}$, then the induced representation acts on $L_2(D)$ as follows:

$$[\rho(g)f](z) = \frac{\left|\bar{\beta}z+\alpha\right|^2}{(\bar{\beta}z+\alpha)^2}\, f\left(\frac{\bar{\alpha}z+\beta}{\bar{\beta}z+\alpha}\right) = \frac{\beta\bar{z}+\bar{\alpha}}{\bar{\beta}z+\alpha}\, f\left(\frac{\bar{\alpha}z+\beta}{\bar{\beta}z+\alpha}\right). \tag{2.28}$$

Since we are not in the unimodular setting now, we calculate the invariant measure on the unit disk to be $(1-|z|^2)^{-2}\,dz\wedge d\bar{z}$. The representation (2.28) is unitary and belongs to the discrete series [46, § IX.3]. In contrast to equivalent representations used in complex analysis, our expression (2.28) has clearer composition formula, cf. [8, (**)]. However, (2.28) does not preserves usual analyticity. We can use either conformal-invariant modification of the Cauchy–Riemann equations [39, § 5.3], or introduce an additional peeling map, which intertwines our representation with the more common one.

3. Covariant Transform

Representation theory is behind many important calculations in analysis, this is illustrated in the present section. The group-theoretical foundations of coherent states/wavelets are well-known and widely appreciated [1, 48]. However, the significance of groups and representations in operator theory [32,35,36,39,43] is still awaiting its recognition.

3.1. **Induced Covariant Transform.** The following definition is a general template, which admits various specialisations adjusted to particular cases.

Definition 3.1. [37] Let ρ be a representation of G in a vector space V. For a vector space U and an operator $F: V\rightarrow U$, the *covariant transform* is the map:

$$[\mathcal{W}_F v](g) = F(\rho(g)v), \qquad v\in V, \quad g\in G, \tag{3.1}$$

to U-valued functions on G. In this context we call F a *fiducial operator*.

An important particular case of the above definition is provided by a linear functional $F\in V^*$, the covariant transform produces *matrix coefficients* of the representation [24, Ex. I.1.2.12]. In the case of a Hilbert space V, such a functional is provided by a pairing with a vector $f\in V$, which is known as *mother wavelet* or *vacuum state* [1,48]. Then, the covariant transform becomes the wavelet transform:

$$\tilde{v}(g) = [\mathcal{W}_f v](g) := \langle \rho(g)v, f\rangle = \langle v, \rho^*(g)f\rangle = \langle v, f_g\rangle\,, \tag{3.2}$$

where $f_g=\rho^*(g)f$ are called wavelets. The image $\mathcal{W}_f v$ is a scalar-valued function. The scalar case is very important, however, it does not cover all interesting situations, see [43] and Example 3.7. We also may require a functional F associated to a singular mother wavelet, i.e. a distribution, cf. [35, § 2.3].

If the representation ρ and the operator F are bounded, then the image of $\mathcal{W}_F$ consists of bounded functions on G. Weak continuity of ρ suffices for continuity of $\mathcal{W}_F v$. An important property of $\mathcal{W}_F$ is as follows.

Lemma 3.2. *The covariant transforms intertwines the left* $\Lambda(g): f(g')\mapsto f(g^{-1}g')$ *and right* $R(g)$ *regular representations of* G *with the following actions of* ρ*:*

$$R(g)\mathcal{W}_F = \mathcal{W}_F\rho(g) \quad \textit{and} \quad \Lambda(g)\mathcal{W}_F = \mathcal{W}_{F\circ\rho(g^{-1})} \quad \textit{for all } g\in G. \tag{3.3}$$

There is the following simple but useful consequence of the above Lemma.

Corollary 3.3. [39, Cor. 5.8] *Let* ρ *be a linear representation of a group* G *on a space* V*, which has an adjoint representation* ρ^* *on the dual space* V^**. Let a mother wavelet* $f\in V^*$ *satisfy the equation*

$$\int_G a(g)\,\rho^*(g)f\,dg = 0,$$

for a fixed distribution $a(g)$ *and a (not necessarily invariant) measure* dg*. Then, any wavelet transform* $\tilde{v} = \langle v, \rho^*(g)f\rangle$ *obeys the following right-invariant condition:*

$$D\tilde{v} = 0, \qquad \textit{where} \quad D = \int_G \bar{a}(g)\, \Lambda(g)\, dg, \tag{3.4}$$

with Λ *being the left regular representation of* G*.*

As we will see below, the above distribution a is often a linear combination of derivatives of the Dirac's delta functions, therefore the operator D turns to be a differential operator. Further examples can be found in [44, Ex. 5.9–11].

Often we need only a part of covariant transform. For a Lie group G and its subgroup H, we fix a continuous section $s : H\backslash G \to G$, which is a right inverse to the projection $p : G \to H\backslash G$.

Definition 3.4. [39, § 5.1] Let $F : V \to U$ intertwine the restriction of ρ to H with a character χ of H: $F(\rho(h)v) = \chi(h)F(v)$ for all $h \in H$, $v \in V$. Then, the *induced covariant transform* is:

$$[\mathcal{W}_F v](x) = F(\rho(s(x))v), \qquad v \in V, \quad x \in H\backslash G. \tag{3.5}$$

Under our assumptions, the induced covariant transform intertwines ρ with the representation induced from H by the character χ. To use the condition (3.4) for the induced covariant transform, we need first apply the lifting $\mathcal{L}_\chi$ (2.18) to $\mathcal{W}_F v$ and then the operator D. A collection of such conditions (3.4) can characterise the image $\mathcal{W}_f V$ among all functions on X, see Examples 3.5(i) and 3.5(iv) below.

In many cases, e.g. for square integrable representations and an admissible mother wavelet $v \in V$, the image space of the covariant transform is a reproducing kernel Hilbert space [1, Thm. 8.1.3]. That means that for any function $v \in \mathcal{W}_f V$ we have the integral reproducing formula:

$$v(y) = \int_X v(x)\, \bar{k}_y(x)\, dx, \tag{3.6}$$

where the *reproducing kernel* k_y provides the twisted convolution with the normalised covariant transform $\mathcal{W}_f(\rho(s(y)^{-1})f)$ for the mother wavelet f, see Cor. 4.8. For a function $v \notin \mathcal{W}_f V$, the right-hand side of (3.6) defines its projection to the space $\mathcal{W}_f V$.

Example 3.5. (i) For $G = \mathbb{H}$, $H = Z$ and the representation $\rho_\hbar$ (2.23) on $L_2(\mathbb{R})$, we have $\rho_\hbar(s, 0, 0) = e^{2\pi i\hbar s}$. Thus any function $f \in L_2(\mathbb{R})$ is suitable for the induced wavelet transform. Explicitly:

$$\begin{aligned}[\mathcal{W}_f v](x, y) = \langle \rho_\hbar(x, y)v, f\rangle &= \int_{\mathbb{R}} e^{\pi i\hbar(2yx'+xy)}\, v(x'+x)\, \bar{f}(x')dx' \\ &= \int_{\mathbb{R}} e^{2\pi i\hbar y x''}\, v(x''+\tfrac{1}{2}x)\, \bar{f}(x''-\tfrac{1}{2}x)\, dx''.\end{aligned} \tag{3.7}$$

The last expression is known as *Fourier–Wigner transform* [12, § 9.2; 17, § 1.4].

For the representation (2.22) and the functional produced by pairing with the Gaussian $\phi(z) = e^{-\pi\hbar|z|^2/2}$ the covariant transform $\mathcal{W}_\phi$ is:

$$\begin{aligned}[\mathcal{W}_\phi f](z) &= \int_{\mathbb{C}} e^{\pi\hbar(z\bar{z}'-\bar{z}z')/2} f(z+z')\, e^{-\pi\hbar|z'|^2/2}\, dz' \wedge d\bar{z}' \\ &= \int_{\mathbb{C}} f(z'')\, e^{\pi\hbar z\bar{z}''}\, e^{-\pi\hbar(|z''|^2+|z|^2)/2}\, dz'' \wedge d\bar{z}'',\end{aligned} \tag{3.8}$$

where $z'' = z + z'$. This is Fock–Segal–Bargmann (FSB) transform, it presents the FSB *reproducing kernel* $k(z, z'') = e^{\pi\hbar z\bar{z}''}\, e^{-\pi\hbar(|z''|^2+|z|^2)/2}$. Note, that the second exponent is usually attributed to the weight [4; 9; 17, § 1.6].

The image $F_2(\mathbb{C})$ of (3.8) is an irreducible invariant subspace of $L_2(\mathbb{C})$ with the corresponding orthogonal projection:

$$P_F : L_2(\mathbb{C}) \to F_2(\mathbb{C}), \tag{3.9}$$

provided by (3.8). The space $F_2(\mathbb{C})$ is characterised by the differential equation $(\partial_{\bar{z}} - z)f = 0$, which follows from Cor. 3.3 with the distribution $a(s, x, y) = \delta'_x(s, x, y) - i\delta'_y(s, x, y)$.

(ii) For an AHW group $\tilde{G}$, H being its centre and the representation (2.25), any function l on G produces a pairing appropriate for the induced covariant transform, cf. (3.7):

$$[\mathcal{W}_l v](g, \chi) = \int_G \chi^k(g')\, f(g+g')\, \bar{l}(g')\, dg', \tag{3.10}$$

where we integrate over the Haar measure on G.

(iii) For $G = \mathbb{D}$, $H = M$, the representation ρ_h (2.26) is induced by a character of the centre C. Thus, any functional can be used for the induced covariant transform to $C\backslash\mathbb{D}$:

$$\begin{aligned}[\mathcal{W}_f w](t, u, v, s, x, y) &= \int_{\mathbb{H}} e^{\pi i h(2s't+st+\frac{1}{4}(x'y-xy')t+(2x'+x)u+(2y'+y)v)} \\ &\quad \times w(s+s'+\tfrac{1}{2}(x'y-xy'), x+x', y+y')\, \bar{f}(s', x', y')\, ds'\, dx', dy' \\ &= \int_{\mathbb{H}} w(s'+\tfrac{1}{2}s+\tfrac{1}{4}(x''y-xy''), x''+\tfrac{1}{2}x, y''+\tfrac{1}{2}y) \\ &\qquad \times \bar{f}(s''-\tfrac{1}{2}s-\tfrac{1}{4}(x''y-xy''), x''-\tfrac{1}{2}x, y''-\tfrac{1}{2}y) \\ &\qquad \times e^{2\pi i h(s''t+x''u+y''v)}\, ds''\, dx''\, dy''.\end{aligned}$$

A similarity with the Fourier–Wigner transform (3.7) is explicit.

(iv) For $G = SU(1,1)$, $H = K$ and the induced representation ρ (2.28), a pairing with the function $l_0(z) = 1 - |z|^2$, has the property

$$\langle \rho(h)v, l_0 \rangle = e^{2i\phi} \langle v, l_0 \rangle, \qquad h = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \in K.$$

Thus, l_0 can be used for the induced covariant transform:

$$\begin{aligned}[][\mathcal{W}_0 v](w) &= \int_D \frac{w\bar{z}+1}{\bar{w}z+1}\, v\left(\frac{z+w}{\bar{w}z+1}\right)\, (1-|z|^2)\, \frac{dz\wedge d\bar{z}}{(1-|z|^2)^2}\\
&=(1-|w|^2)\int_D \frac{v(\zeta)}{(1-\bar{\zeta}w)^2}\,\frac{d\zeta\wedge d\bar{\zeta}}{1-|\zeta|^2}, \qquad \text{where } \zeta=\frac{z+w}{\bar{w}z+1}.
\end{aligned} \tag{3.11}$$

Up to the factor $\frac{1-|w|^2}{1-|\zeta|^2}$ discussed in Example 2.2(iv), this is known as the *Bergman integral* [8]. The image space $B_2(\mathbb{D})$ of $\mathcal{W}_0$ is SU(1,1)-invariant subspace of $L_2(\mathbb{D})$, which is called *Bergman space*. The orthogonal projection:

$$P : L_2(\mathbb{D}) \to B_2(\mathbb{D}), \tag{3.12}$$

presented by the Bergman integral (3.11) is called the Bergman projection. On $B_2(\mathbb{D})$ the integral (3.11) acts as a reproducing formula, cf. (3.6).

The Bergman space is in the kernel of the differential operator $\frac{z}{1-|z|^2} - \partial_{\bar{z}}$. For an expression of this operator in terms of SU(1,1) and Cor. 3.3 see [33, Ex. 3.7(a); 46, § IX.5].

3.2. **Berezin Covariant Symbol.** An important observation [35] is that, for a representation ρ of G in a vector space V, we have a representation

$$\hat{\rho}(g_1, g_2) : A \mapsto \rho(g_2)^{-1} A \rho(g_1), \qquad (g_1, g_2) \in G \times G \tag{3.13}$$

of $G \times G$ on the space $B(V)$ of bounded linear operators on V.

Definition 3.6. [39, § 4.3; 43] For a fixed operator $F : B(V) \to U$ the *covariant symbol* $\tilde{A}(g_1, g_2)$ is the covariant transform defined by the representation $\hat{\rho}$ and the operator F:

$$\tilde{A}(g_1, g_2) = F(\hat{\rho}(g_1, g_2)A) = F(\rho(g_2)^{-1} A \rho(g_1)), \qquad \text{where } (g_1, g_2) \in G \times G. \tag{3.14}$$

We also use the notation $\tilde{A}(g)$ for $\tilde{A}(g, g)$.

Since the covariant symbol is a special case of the covariant transform, the respective variants for the scalar case and induced form are applicable as well. The combination of both has the special name. For fixed $f \in V$ and $l \in V^*$, the *Berezin covariant symbol* $\tilde{A}(x_1, x_2)$ is the induced covariant transform defined by the representation $\hat{\rho}$ and the functional $F(A) = l(Af)$:

$$\tilde{A}(x_1, x_2) = F(\hat{\rho}(s(x_1), s(x_2))A) = l(\rho(\rho(s(x_2)^{-1}) A s(x_1))f), \tag{3.15}$$

where $x_1, x_2 \in H\backslash G$. Again, we denote $\tilde{A}(x) = \tilde{A}(x, x)$.

As before, this definition is most useful if f and l are eigenvectors for all transformations $\rho(h)$, $h \in H$. An important particular case of the construction is a unitary representation in a Hilbert space V and the functional $l \in V^*$ be a pairing with $f \in V$ [5, § 1.2]:

$$\tilde{A}(x, y) = \left\langle \rho(s(y))^{-1} A \rho(s(x)) f, f \right\rangle = \left\langle A\rho(s(x))f, \rho(s(y))f \right\rangle = \left\langle Af_x, f_y \right\rangle, \tag{3.16}$$

where $f_x = \rho(s(x))f$, $f_y = \rho(s(y))f$.

Example 3.7. There is a large variety of possibilities (even for a fixed group G) provided by a selection of various subgroups H, representations ρ and fiducial functionals F. We will illustrate this for the Heisenberg group. Note that, our list

is based on the most popular options and is far from being exhausting. For other groups, the number of possibilities is not smaller.

(i) For the Heisenberg group, to make a structure of the listed options we introduce a subdivision.

(a) For $G = \mathbb{H}$ and the representation (2.23), take $f(y) = l(y) = \delta(y)$—the Dirac delta function. For the subgroup H_x (2.9) and the homogeneous space $\mathbb{R} = H_x \backslash \mathbb{H}$ representation (2.23) acts on $[\rho(s(0,-x,0)\delta](x') = \delta_x(x') = \delta(x'-x)$. Consider a smoothing operator $A : S' \rightarrow S$, where S is the *Schwartz space* of smooth rapidly decreasing functions on the real line and S' is its dual—the space of tempered distributions. Then the Berezin covariant symbol is:

$$\tilde{A}(x_1, x_2) = \langle A\delta_{x_1}, \delta_{x_2} \rangle \,, \tag{3.17}$$

which will be related to the Schwartz kernel below.
The reader may notice that our usage of the Heisenberg group looks excessive in this case: shifts on the real line are completely sufficient. Thus, we are moving to the next case.

(b) Again consider $G = \mathbb{H}$ this time with the subgroup H_x (2.9) and the analogous subgroup $H_y = \{(s, 0, x) \in \mathbb{H}\}$. Accordingly, for the representation (2.23) we take $f(x) \equiv 1$ and $l(x) = \delta(x)$ both being tempered distributions from S'. Then, $[\rho_\hbar(0,0,y)f](x') = e^{\pi i \hbar 2x' y} f(x')$ and $[\rho_\hbar(0,-x,0)\delta](x') = \delta(x'-x)$. Since, the Fourier transform of $\rho_\hbar(0,0,y)f$ is the delta function δ_y, for the PDO A_{KN} (1.2) with a smooth symbol a, the Berezin symbol

$$\tilde{A}_{KN}(y, x) = a(x, y),$$

is its Kohn–Nirenberg symbol a.

(c) For $G = \mathbb{H}$, $H = Z$, the representation (2.22) and the both l and f be the Gaussian $\phi(z) = e^{-\pi\hbar|z|/2}$, the transformation (3.16) is the Wick (or Berezin) symbol of an operator A [4; 9; 17, § 2.7; 22]. The simplest calculation of the covariant symbol can be performed for the *Toeplitz operator* $T_a = P_F a P_F$, with $a \in L_\infty(\mathbb{C})$ and P_F (3.9). For the Gaussian ϕ and $\phi_z = \rho^F_\hbar(s(z))\phi$ we found:

$$\begin{aligned}
\tilde{T}_a(w, z) &= \langle T_a \phi_w, \phi_z \rangle = \langle P_F a \phi_w, \phi_z \rangle = \langle a\phi_w, P_F^* \phi_z \rangle = \langle a\phi_w, \phi_z \rangle \\
&= \int_{\mathbb{C}} a(z')\, e^{-\pi\hbar(\bar{w}z' + |w|^2/2 + |z'|^2/2)} e^{-\pi\hbar(z\bar{z}' + |z|^2/2 + |z'|^2/2)}\, dz' \wedge d\bar{z}' \\
&= e^{-\pi\hbar(|w|^2/2 + |z|^2/2)} \int_{\mathbb{C}} a(z')\, e^{-\pi\hbar(\bar{w}z' + z\bar{z}' + |z'|^2)}\, dz' \wedge d\bar{z}'.
\end{aligned} \tag{3.18}$$

Clearly, $\tilde{T}_a(z, z)$ is not much different from the FSB transform (3.8) of a.
It is worth to notice, that the unitary equivalent model on the real line appears if both f and l are the Gaussians $e^{-\pi x^2/2}$ on the real line. The respective contravariant symbol translates to the language of quantum mechanics as the transition amplitude of a quantum mechanical observable (in the Schrödinger model) between states with minimal uncertainty.

Another class of operators with a useful Berezin calculus are *composition operators* [8], i.e. an operator $C_\phi : f \mapsto f \circ \phi$ for a fixed map $\phi : X \to X$ of the domain to itself.

(d) There is another approach for $G = \mathbb{H}$ and the representation (2.23). We take an (operator-valued) fiducial operator $F : B(L_2(\mathbb{R})) \to B_s(L_2(\mathbb{R}))$, where $B_s(L_2(\mathbb{R}))$ is the space of bounded shift-invariant operators on $L_2(\mathbb{R})$. F is defined by:

$$F : A \mapsto A_0, \qquad \text{such that} \quad \lim_{\delta\to 0} |||M_\delta A M_\delta - A_0||| = 0, \tag{3.19}$$

where M_δ is an operator of multiplication by the indicator function of δ-neighbourhood of the origin and $|||\cdot|||$ denotes the essential norm (modulo compact operators). The limit exists for *operators of local type* [52].

In particular, for the operator M_f of multiplication by a function $f(x)$ we have $FM_f = f(0)I$. Therefore, for the representation $\rho_\hbar$ (2.23), we obtain the eigenfunction property $F\rho_\hbar(s,0,y) = e^{2\pi i\hbar s}I$ for all $(s,0,y) \in H_x$ (2.9). Thus, we can use the induced form of the covariant symbol (3.14) only for values $\rho_\hbar(0,x,0)$, where $x \in \mathbb{R} = H_x\backslash\mathbb{H}$ (2.23)—they are shifts on the real line. Thus the localisation map (3.19) defines the covariant transform $A_x = F(\rho_\hbar(0,x,0)A\rho_\hbar(0,-x,0))$—the local representative of the operator A at a point x [43,52].

Another important example of operators of local type are SIOs—convolutions on $\mathbb{R}$ with singular kernels—moreover, $F(S) = S$ for any SIO S with a homogeneous kernel. This recovers Simonenko's localisation technique for the calculus of operators generated by SIOs and operators of multiplications [43,52,53].

(ii) For an AHW group $\tilde{G}$, we can essentially repeat all approaches for $\mathbb{H}$ described above. For example, we provide an analogue of 3.7(i)(b). For $\tilde{G}$ generated by a commutative group G consider subgroups H_G (2.11) and the similar subgroup $H_{\hat{G}} = \{(z,g,0)\}$, the respective homogeneous spaces are $G = H_G\backslash G$ and $\hat{G} = H_{\hat{G}}\backslash G$. Take $f(g) \equiv 1$ on G and $l(g) = \delta(g)$. For the representation (2.25), we have $[\rho_1(1,0,\chi)f](x') = \chi(x')$ and $[\rho_1(1,-x,1)\delta](x') = \delta_x(x') = \delta(x'-x)$. Then, the Berezin symbol of the operator A (1.5) is:

$$\tilde{A}(x,\chi) = \langle A\chi, \delta_x\rangle = a(x,\chi),$$

i.e. the symbol a entering the integral (2.25). Other variations of 3.7(i)(a)–(i)(d) can be obtained in similar ways.

(iii) For $G = \mathbb{D}$, $H = M$ and the representation (2.26), we can use the localisation approach from 3.7(i)(d). For a localisation functional F at the origin of $\mathbb{H}$ similar to (3.19), we calculate $F(\rho_h(z,t,u,v,0,0,0)) = e^{2\pi i h z}I$ for the representation (2.26) and $(z,t,u,v,0,0,0)$ in the subgroup M (4.8). Thus, it is sufficient to perform the covariant transform (3.14) for $[\rho_h(0,0,0,0,s,x,y)$, which are shifts on $\mathbb{H}$. In this way we recovered the calculus of SIO on the Heisenberg group initiated in [13,14], see also [28,29,31]. This can be extended to more general nilpotent Lie groups. To this end we need to consider a suitable group of dilations, which acts by automorphisms

of the nilpotent group [16, § 1.A]. Such a covariant calculus was recently considered in [43].

(iv) For $G = SU(1,1)$, $H = K$ and the representation (2.28) we can follow the suit of 3.7(i)(c) by setting $f(z) = l(z) = 1 - |z|^2$. The Berezin symbol (3.16) is well-known [6] and very important in the theory of operators [8; 47, § B.4.1.8; 59, § A.3]. Similarly to the Heisenberg group, the simplest calculation of the covariant symbol appear for the *Toeplitz operator* $T_a = PaP$, where $a(z) \in L_\infty(\mathbb{D})$ and P is the Bergman projection (3.12). Using expressions from Examples 2.1(iv) and 2.2(iv), for the $l_0(\zeta) = 1 - |z|^2$ we calculate:

$$l_w(\zeta) = [\rho(s(z))l_0](\zeta) = \frac{(1-|w|^2)(1-|\zeta|^2)}{(1+\bar{w}\zeta)^2}.$$

Then:

$$\begin{aligned}\tilde{T}_a(w,z) &= \langle T_a l_w, l_z\rangle = \langle Pal_w, l_z\rangle = \langle al_w, P^* l_z\rangle = \langle al_w, l_z\rangle \\ &= \int_{\mathbb{D}} a(\zeta)\frac{(1-|w|^2)(1-|\zeta|^2)}{(1+\bar{w}\zeta)^2}\overline{\left(\frac{(1-|z|^2)(1-|\zeta|^2)}{(1+\bar{z}\zeta)^2}\right)}\frac{d\zeta\wedge d\bar{\zeta}}{(1-|\zeta|^2)^2} \\ &= (1-|w|^2)(1-|z|^2)\int_{\mathbb{D}}\frac{a(\zeta)}{(1+\bar{w}\zeta)^2(1+z\bar{\zeta})^2}\,d\zeta\wedge d\bar{\zeta}. \end{aligned} \tag{3.20}$$

Another opportunity to investigate operators on the Bergman space is the localisation technique similar to 3.7(i)(d). The localisation can be combined with the Berezin calculus [59].

3.3. **Calculus of Covariant Symbols.** If a functional F and a representation ρ are both linear, then the resulting covariant transform $\mathcal{W}_F$ (3.1) is a linear map. If $\mathcal{W}_F$ is injective, e.g. due to irreducibility of ρ, then $\mathcal{W}_F$ transports a norm $\|\cdot\|$ existing on V to a norm $\|\cdot\|_F$ on the image space $\mathcal{W}_F V$ by the simple rule [44]:

$$\|u\|_F := \|v\|, \qquad \text{where the unique } v \in V \text{ is defined by } u = \mathcal{W}_F v. \tag{3.21}$$

By the very definition, $\mathcal{W}_F$ is an isometry $(V, \|\cdot\|) \to (\mathcal{W}_F V, \|\cdot\|_F)$. Moreover, if the representation ρ acts on $(V, \|\cdot\|)$ by isometries then $\|\cdot\|_F$ is right invariant due to Lem. 3.2.

In most cases, the transported norm can be naturally expressed in the original terms for G. For example, for a square integrable modulo a subgroup H representation ρ and an admissible mother wavelet $f \in V$ the transported by (3.2) norm coincides with the L_2-norm on $X = H\backslash G$. Explicitly, for $v_{1,2} \in V$ and $\tilde{v}_{1,2}(x) = \langle v_{1,2}, \rho(s(x))f\rangle_V$ [1, Ch. 8]:

$$\langle v_1, v_2\rangle_V = \langle \tilde{v}_1, \tilde{v}_2\rangle_{\mathcal{W}}, \quad \text{where } \langle \tilde{v}_1, \tilde{v}_2\rangle_{\mathcal{W}} = \int_X \tilde{v}_1(x)\,\overline{\tilde{v}_2(x)}\,dx. \tag{3.22}$$

Another example of a transported norm is the norm on the Hardy space in the half-plane [44].

The particular case of the above transportation is provided by the Berezin transform. For an operator A on a normed space V, the norm of A has the standard definition: $\|A\| = \sup_{\|v\|\leqslant 1}\|Av\|$. For an isometric representation ρ of G on V and $\|f\| \leqslant 1$ and $\|l\| \leqslant 1$, the associated Berezin transform $\tilde{A}(x,y)$ (3.16) is a function on $X \times X$ bounded by $\|A\|$. The opposite statement—boundedness

of $\tilde{A}(x,y)$ implies boundedness of A—is a variation of the *reproducing kernel thesis (RKT)* [47, § B.4.1.8]. Another related topic is a connection of compactness of A and vanishing of $A(x,y)$ "near to the boundary" [8,9]. We return to RKT in Subsection 4.3.

The isometric property (3.22) allows us to follow [5, § 1.2] and deduce composition rule for Berezin covariant symbols [35, Prop. 3.2]:

$$\begin{aligned}\widetilde{AB}(x,y) &= \langle ABf_x, f_y\rangle_V = \langle Bf_x, A^*f_y\rangle_V\\ &= \int_X \widetilde{Bf_x}(z)\,\overline{\widetilde{A^*f_y}(z)}\,dz = \int_X \langle Bf_x, f_z\rangle\,\langle f_z, A^*f_y\rangle\,dz\\ &= \int_X \langle Bf_x, f_z\rangle\,\langle Af_z, f_y\rangle\,dz = \int_X \tilde{B}(x,z)\,\tilde{A}(z,y)\,dz.\end{aligned}\tag{3.23}$$

One can note, that covariant symbols behaves (up to the order of A and B in the last integral) like integral kernels and this is not a simple coincidence, see below. Our formula is more straightforward than the original [5, § 1.2] since we do not need a normalization.

Example 3.8. (i) For the Heisenberg group in the setup of 3.7(i)(a), the fiducial functional of pairing with $f = \delta$ produces the identity operator in $\tilde{v}(x) = \langle v, f_x\rangle$ (3.2). Since we have the isometry (3.22) in the trivial way, the composition rule (3.23) follows. Keeping in mind that the Berezin covariant transform (3.17) is the Schwartz kernel with reversed arguments, we obtained the well-known integral formula for the composition of Schwartz kernels.

In the setup of 3.7(i)(c), the covariant transform turns to be a reproducing formula (3.8) on the FSB space, thus, is an isometry. The specialisation of the composition rule (3.23) for the Toeplitz operators in the FSB space can be found in many works starting from [4].

(ii) For an AHW group $\tilde{G}$, we can essentially repeat all approaches (Schwartz kernel, PDO-type, Toeplitz operators and localisation techniques) which are in use for the Heisenberg group with respective norms and compositions formulae.

(iii) For the Dynin group $\mathbb{D}$ and the representation (2.26), we recall the localisation context from 3.7(i)(d) and 3.7(iii). Let $P_{0,\delta}$ be the projection provided by multiplication with the characteristic function of δ-neighbourhood of $0 \in \mathbb{H}$. Then, the representation (2.26) produces similar projections $P_{g,\delta}$ for an arbitrary $g \in \mathbb{H}$. For an operator A on $L_2(\mathbb{H})$ we can build a Berezin covariant symbol $\tilde{A}_\delta(g_1,g_2) = P_{g_2,\delta}AP_{g_1,\delta}$. If A is an operator of local type [52], then $\tilde{A}_\delta(g_1,g_2)$ is a compact for all $\delta < |g_1 - g_2|$. Thus, modulo compact operators the symbol $\tilde{A}(g_1,g_2) = \lim_{\delta\to 0}\tilde{A}_\delta(g_1,g_2)$ vanishes outside of the diagonal. Therefore, the covariant symbol becomes a field of local representatives $\tilde{A}(g) = \tilde{A}(g,g)$, $g \in \mathbb{H}$ [14,28]. The isometry (3.21) becomes $|||A||| = \sup_g \left\|\tilde{A}(g)\right\|$. The composition rule (3.23) reduces to point-wise multiplication of local representatives: $\widetilde{AB}(g) = \tilde{A}(g)\tilde{B}(g)$.

(iv) For $G = SU(1,1)$, $H = K$ and the representation (2.28) we also have the reproducing formula (3.11) on the Bergman space. The respective

composition formula for Toeplitz operators is well-known [6, § 4.2; 8; 59, § A.3].

4. Relative Convolutions

4.1. **Integrated Representations and Contravariant Symbols.** Let G be a locally compact group, a left-invariant (Haar) measure on G is denoted by dg. Let ρ be a representation of the group G in a vector space V. The representation can be extended to a function k on G though integration

$$\rho(k) = \int_G k(g)\, \rho(g)\, dg. \tag{4.1}$$

In the simplest case k has scalar values, however, the same formula is meaningful for functions with values in operators on the representation space V.

The integral (4.1) can be defined in a weak sense for various combinations of functions and representations. One of the natural setups is a bounded (e.g. unitary) representation ρ and a summable function k. In this case we obtain a homomorphism of the convolution algebra $L_1(G, dg)$ to an algebra of bounded operators on V:

$$\rho(k_1)\rho(k_2) = \rho(k_1 * k_2), \qquad \text{where} \quad [k_1 * k_2](g) = \int_G k_1(g_1)\, k_2(g_1^{-1} g)\, dg_1.$$

For a representation ρ induced from a subgroup H, all operators $\rho(h)$, $h \in H$ act in (2.20) locally. That becomes especially trivial if $\rho(h)$ are scalars. Thus, for induced representations, we are mainly interested in the "complement" $H\backslash G$ in the expression (4.1). For a continuous section $s : H\backslash G \to G$, we rewrite (4.1) to become an operator of a *relative convolution* [34]:

$$\rho(k) = \int_X k(x)\, \rho(s(x))\, dx, \tag{4.2}$$

with a kernel k defined on $X = H\backslash G$ with a (quasi-)invariant measure dx (2.15). Again, the most natural domain of this definition is a bounded representation ρ and a summable k from $L_1(X, dx)$. Furthermore, in many cases we need to (and can) extend meaning of (4.2) for suitable functions and distributions, e.g. the Dirac delta function and its derivatives.

Example 4.1. (i) We already mentioned that relative convolutions generated by the Schrödinger representation (1.8) of the Heisenberg group are PDO (1.9). In this case $G = \mathbb{H}$ and $H = \{(s, 0, 0) \mid s \in \mathbb{R}\}$—the centre of $G = \mathbb{H}$. It is the original inspiration for this approach [17, 22, 34].

(ii) For the AHW group $\tilde{G}$ and its representation (2.25) with $k=1$, take a function $\sigma(g,\chi)$ on $X=\mathbb{T}\backslash\tilde{G}=G\times\hat{G}$ and calculate, cf. [17, (2.32)]:

$$\begin{aligned}
[\rho(a)f](g') &= \int_G \int_{\hat{G}} \sigma(g,\chi)\,\chi(g')\,f(g+g')\,dg\,d\chi \\
&= \int_G \hat{\sigma}_2(g,g')\,f(g+g')\,dg \\
&= \int_G \int_{\hat{G}} \hat{\sigma}(\xi,g')\,\bar{\xi}(g)\,f(g+g')\,dg\,d\xi \\
&= \int_G \int_{\hat{G}} \hat{\sigma}(\xi,g')\,\bar{\xi}(g''-g')\,f(g'')\,dg''\,d\xi \\
&= \int_{\hat{G}} \xi(g')\hat{\sigma}(\xi,g') \int_G f(g'')\,\bar{\xi}(g'')\,dg''\,d\xi, \qquad (4.3)
\end{aligned}$$

here $g''=g+g'$, $\hat{\sigma}_2$ is a function on $G\times G$, which is the Fourier transform of σ in second variable. The last expression (4.3) coincides with Kohn–Nirenberg type PDO (1.5) for $a(g,\xi)=\hat{\sigma}(\xi,g)$, cf. [50, Part II].

(iii) For the Dynin group $\mathbb{D}$, the unitary representation ρ (2.26) on $L_2(\mathbb{H})$ is obtained from its infinitesimal action (2.1) and (2.5). The integrated representation (4.1) was considered in [13] as a generalisation of the Weyl quantization from $\mathbb{H}$ to the group $\mathbb{D}$.

If function k has the structure $k(z,t,u,v,s,x,y)=\delta(z,t,u,v)k_1(s,x,y)$, where δ is the Dirac delta function, then $\rho(k)$ is a convolution on the Heisenberg group with the kernel k_1. On the other hand, if $k(z,t,u,v,s,x,y)=\delta(z)k_2(t,u,v)\delta(s,x,y)$, then $\rho(k)$ is an operator of multiplication by $\hat{k}_2(s,x,y)$—the (Euclidean) Fourier transform $(t,u,v)\to(s,x,y)$ of k_2. Thus, the integrated representation (4.1) in this case belongs to the algebra of operators generated by convolutions on the Heisenberg group and multiplications by functions, which were investigated, for example, in [13,14,28,43]. For a suitable choice of symbols, this operators coincide with (1.6) used in [3].

Furthermore, we can observe that in both cases kernels depend on the coordinate z through the delta function. Thus, instead of the integrated representation (4.1) we can use the relative convolutions (4.2) for $G=\mathbb{D}$ and H being its centre, cf. the case of the Heisenberg group above.

(iv) For $G=SU(1,1)$ and $H=K$, a substitution of (2.14) into representation (2.28) produces the relative convolutions:

$$\begin{aligned}
[\rho(k)v](z) &= \int_D k(w)\,\frac{w\bar{z}+1}{\bar{w}z+1}\,v\left(\frac{z+w}{\bar{w}z+1}\right)\frac{dw\wedge d\bar{w}}{(1-|w|^2)^2} \\
&= \int_D \frac{1-z\bar{\zeta}}{1-\bar{z}\zeta}\,k\left(\frac{z-\zeta}{\bar{z}\zeta-1}\right)v(\zeta)\frac{d\zeta\wedge d\bar{\zeta}}{(1-|\zeta|^2)^2}, \quad \text{where } w=\frac{z-\zeta}{\bar{z}\zeta-1}. \qquad (4.4)
\end{aligned}$$

Interestingly, the last integral can also be interpreted as $\tilde{v}(z)=\langle v,\rho(s^{-1}(z))\bar{k}\rangle$, which is the induced wavelet transform on $K\backslash SU(1,1)$ [39, § 5.5] with the mother wavelet $\bar{k}$.

The indicated connection of relative convolutions with the induced wavelet transform is not an exception. It occurs in many other cases when the representation space V consists of functions defined on the homogeneous space $X=H\backslash G$, e.g. the FSB space of analytic functions on $\mathbb{C}^n=Z\backslash\mathbb{H}^n$. In general, the covariant

transform, the Berezin symbol, integrated representations and the contravariant symbol (considered below) are closely connected and, sometimes, even confused.

4.2. **Twisted Convolutions.** It is desirable to have an efficient symbolic calculus of relative convolutions. For exponential Lie groups, a calculus in terms of the respective Lie algebras was initiated in [34]. However, the exponential property is rather restrictive, for example, $SU(1,1)$ does not posses it. Here we provide another algebraic condition, which is sufficient for relative convolutions to be closed under multiplication.

In the notations of Section 2.2, for any x_1, $x_2 \in X = H\backslash G$ there is the unique $x \in X$ defined by the identity

$$s(x_1)\, s(x_2) = hs(x), \qquad \text{that is} \quad x = p(s(x_1)\, s(x_2)) = x_1 \cdot s(x_2), \tag{4.5}$$

where the last expression uses notation (2.7).

The relation (4.5) defines a binary operation $(x_1, x_2) \mapsto x$, which turns X into a semigroup. It is not a group unless H is a normal subgroup of G. One can develop a separate theory for semigroups from homogeneous spaces, for example, in [58] they are called *gyrogroups*. However, we prefer to proceed in terms of the original group G and its subgroup H.

For given x_2, $x \in X$, there is the only $x_1 = x \cdot (s(x_2))^{-1} \in X$ satisfying the first identity in (4.5). We will use the abbreviation $x_1 = xx_2^{-1}$ for it.

Furthermore, using the transformation rule (2.16) of the measure dx on X we calculate:

$$dx_1\, dx_2 = \frac{\Delta_H(h(x,x_2))}{\Delta_G(h(x,x_2))}\, dx_2\, dx, \quad \text{where} \quad h(x,x_2) = s(x)s^{-1}(x_2)s^{-1}(x \cdot s^{-1}(x_2)).$$

Here, for simplicity, we write $s^{-1}(y)$ instead of the more correct expression $(s(y))^{-1}$.

Let two relative convolutions be defined by scalar-valued summable kernels k_1, $k_2 \in L_1(X)$. Then, starting with the Fubini theorem we calculate:

$$\begin{aligned}
\rho(k_1)\rho(k_2) &= \int_X k_1(x_1)\, \rho(s(x_1))\, dx_1 \int_X k_2(x_2)\, \rho(s(x_2))\, dx_2 \\
&= \int_X \int_X k_1(x_1)\, k_2(x_2)\, \rho(s(x_1)s(x_2))\, dx_1\, dx_2 \\
&= \int_X \int_X k_1(x_1)\, k_2(x_2)\, \rho(s(x_1)s(x_2))\, dx_1\, dx_2 \\
&= \int_X \int_X k_1(xx_2^{-1})\, k_2(x_2)\, \rho(h^{-1}(x,x_2)s(x))\, \frac{\Delta_H(h(x,x_2))}{\Delta_G(h(x,x_2))}\, dx_2\, dx \\
&= \int_X \int_X k_1(xx_2^{-1})\, k_2(x_2)\, \rho(h^{-1}(x,x_2))\, \frac{\Delta_H(h(x,x_2))}{\Delta_G(h(x,x_2))}\, dx_2\, \rho(s(x))\, dx \\
&= \int_X \int_X k_1(xx_2^{-1})\, k_2(x_2)\, \rho(h^{-1}(x,x_2))\, \frac{\Delta_H(h(x,x_2))}{\Delta_G(h(x,x_2))}\, dx_2\, \rho(s(x))\, dx \\
&= \int_X k(x)\, \rho(s(x))\, dx, \qquad (4.6)
\end{aligned}$$

where

$$k(x) = \int_X k_1(xx_2^{-1})\, k_2(x_2)\, \rho(h^{-1}(x,x_2))\, \frac{\Delta_H(h(x,x_2))}{\Delta_G(h(x,x_2))}\, dx_2. \tag{4.7}$$

Note that, if the representation ρ is induced by a character χ of the subgroup H, then $\rho(h^{-1}(x,x_2)) = \chi(h^{-1}(x,x_2))$. Thus, the above integral is scalar valued.

Definition 4.2. For two summable functions k_1 and k_2 on $X = H\backslash G$, their *twisted convolution* $k = k_1 \natural k_2$ is a function k, such that the relative convolution $\rho(k)$ (4.2) equal to the composition $\rho(k_1)\rho(k_2)$ of relative convolutions with the kernels k_1 and k_2, i.e.:

$$\rho(k_1 \natural k_2) = \rho(k_1)\rho(k_2).$$

Then, the result of calculations (4.6) can be encapsulated in the statement:

Proposition 4.3. *The twisted convolutions* $k = k_1 \natural k_2$ *of two functions* k_1 *and* k_2 *on* $H\backslash G$ *is presented by* (4.7).

Remark 4.4. Integrated representations (4.1) and relative convolutions (4.2) map functions to operators. It is a fashion now to call any such map a "quantization". An opposite procedure, e.g. the covariant transform, maps an operator to a function—a symbol of the operator. This can be called "dequantization", respectively. Thus our Defn. 4.2 can be stated in quasi-quantum language as follows: quantize kernels to operators, compose operators, dequantize the composition to the kernel. In this setup the twisted convolution is also known as a *star product*. We refer to [2] for further discussion, references and a more explicit formula in the case of square integrable representations. We also note, that a search of a compatible star product for an arbitrary Poisson manifold is the topic of *deformation quantization*.

Example 4.5. (i) The Heisenberg group $\mathbb{H}$ and its centre Z are unimodular, thus $\Delta_{\mathbb{H}} \equiv 1$ and $\Delta_Z \equiv 1$. Using maps p and s from Example 2.1(i) for $\mathbb{R}^2 = Z\backslash\mathbb{H}$, we calculate:

$$\begin{aligned}(x,y)(x_2,y_2)^{-1} =& (x-x_2, y-y_2),\\ h((x,y),(x_2,y_2)) =& (\tfrac{1}{2}(x_2y - yx_2), 0, 0).\end{aligned}$$

Thus, for a representations (2.21) and (2.23) induced by a character $\chi_\hbar(s,0,0) = e^{2\pi i\hbar s}$, the respective twisted convolution is:

$$(k_1 \natural k_2)(x,y) = \int_{\mathbb{R}^2} k_1(x-x_2, y-y_2)\, k_2(x_2,y_2)\, e^{-\pi i\hbar(x_2y-yx_2)}\, dx_2\, dy_2.$$

This operation is the key for the whole calculus of PDO as explained in [17, § 2.3; 22, § 2]. It is also known as the Groenewold–Moyal star product [2, § 6.1].

(ii) For an AHW group $\tilde{G}$, we calculate $h((g,\chi),(g_2,\chi_2)) = \chi_2(g_2 - g)$. From unimodularity of $\tilde{G}$, the twisted convolution is:

$$(k_1 \natural k_2)(g,\chi) = \int_G \int_{\hat{G}} k_1(g-g_2, \chi-\chi_2)\, k_2(g_2,\chi_2)\, \chi_2(g_2-g)\, dg_2\, d\chi_2.$$

This again looks like a convolution on the Cartesian product $G \times \hat{G}$ with a "twist". A composition with the Fourier transform on $G \times \hat{G}$ maps our twisted convolution to the star product from [61, § 3].

(iii) According to Kirillov's theory, any unitary irreducible representation of a nilpotent group Lie group is induced by a character of the group's centre

C [25, § 15]. Thus, the relative convolution for $C\backslash\mathbb{D}$ is not much different from the whole integrated representation.

On the other hand, since the subgroup M (4.8) is normal then the twisted convolution for $M\backslash\mathbb{D} = \mathbb{H}$ reduces to the group convolution on the Heisenberg group. Some interesting options are located between the extremes C and M. For example, since the Lie algebra of $\mathbb{D}$ is generated by X, Y, T, it is worth to consider twisted convolution associated to the subgroup

$$M' = \{(z, 0, u, v, 0, 0, 0) \in \mathbb{D} \mid (z, u, v) \in \mathbb{R}^3\}. \tag{4.8}$$

(iv) The group $G = SU(1, 1)$ is not unimodular, however $\Delta_G(k) \equiv 1$ for all k in the maximal compact subgroup K [46, § III.1]. The subgroup K is unimodular since it is commutative. Furthermore, using maps (2.14) we calculate:

$$z \cdot s^{-1}(w) = \frac{z - w}{1 - z\bar{w}}, \qquad h(z, w) = \frac{|1 - \bar{z}w|}{1 - \bar{z}w}.$$

Thus, for the representation (2.28) induced from K, the respective twisted convolution is:

$$(k_1 \natural k_2)(z) = \int_D k_1 \left(\frac{z - w}{1 - z\bar{w}}\right) k_2(w) \frac{1 - \bar{z}w}{1 - z\bar{w}} \frac{dw \wedge d\bar{w}}{(1 - |w|^2)^2}.$$

This corresponds to the composition of Berezin's contravariant symbols (see below) and nicely complements the well-known calculus of Berezin's covariant (Wick) symbols considered in Example 3.8(iv) and [6, § 4.2; 8; 59, § A.3].

4.3. **Contravariant Symbol and Toeplitz Operators.** There is a notion immediately derived from the integrated representations: the *contravariant* (aka inverse covariant) transform [37–39, 44]. For an integrated representation ρ (4.1) (or (4.2)) and a fixed vector $w \in V$ the associated contravariant transform of a function k on G (or $X = H\backslash G$) is

$$\mathcal{M}^\rho_w(k) = \rho(\check{k})w, \qquad \text{where} \quad \check{k}(g) = k(g^{-1}). \tag{4.9}$$

The contravariant transform $\mathcal{M}^\rho_w$ intertwines the right regular representation R on $L_2(G)$ and ρ:

$$\mathcal{M}^\rho_w\, R(g) = \rho(g)\, \mathcal{M}^\rho_w. \tag{4.10}$$

Combining with (3.3), we see that the composition $\mathcal{M}^\rho_w \circ \mathcal{W}^\rho_v$ of the covariant and contravariant transform intertwines ρ with itself. We can use the Schur's lemma [1, Lem. 4.3.1; 25, Thm. 8.2.1] to deduce that:

Proposition 4.6. *For an irreducible ρ, the composition $\mathcal{M}^\rho_w \circ \mathcal{W}^\rho_v$ is a multiple $\theta(w, v)I$ of the identity operator. Moreover, the factor $\theta(w, v)$ is a sesquilinear form of vectors w, $v \in V$.*

The following interesting consequence is known in slightly different form for the case of the Heisenberg group [17, (1.47)].

Corollary 4.7. *Assume that the integrated representation ρ is faithful on the image space $\mathcal{W}_{v_2}V$. Then the twisted convolution of wavelet transforms is:*

$$\mathcal{W}_{v_1}u_1 \natural \mathcal{W}_{v_2}u_2 = \theta(u_2, v_1)\mathcal{W}_{v_2}u_1 \tag{4.11}$$

Proof. We note another form $\rho(\mathcal{W}_v u)w = \theta(w,v)u$ of the identity $\mathcal{M}_w^\rho \circ \mathcal{W}_v^\rho = \theta(w,v)I$. Then:

$$\begin{aligned}\mathcal{M}_{v_2}(\mathcal{W}_{v_1}u_1 \natural \mathcal{W}_{v_2}u_2) &= \rho(\mathcal{W}_{v_1}u_1 \natural \mathcal{W}_{v_2}u_2)v_2\\ &= \rho(\mathcal{W}_{v_1}u_1)\rho(\mathcal{W}_{v_2}u_2)v_2\\ &= \rho(\mathcal{W}_{v_1}u_1)\theta(v_2,v_2)u_2\\ &= \theta(v_2,v_2)\rho(\mathcal{W}_{v_1}u_1)u_2\\ &= \theta(v_2,v_2)\theta(u_2,v_1)u_1.\end{aligned}$$

Since the representation ρ is faithful on the image space $\mathcal{W}_{v_2}V$, the obtained result implies (4.11). □

The following particular case of (4.11) is of special interest:

Corollary 4.8. *Under the assumptions of the previous Corollary, the image space $\mathcal{W}_f V$ is reproducing kernel space with the following realisation of a reproducing formula:*

$$\tilde{v} = \tilde{f} \natural \tilde{v}, \qquad \textit{for} \quad \tilde{f} = \mathcal{W}_f f \quad \textit{and any} \quad \tilde{w} \in \mathcal{W}_f V. \tag{4.12}$$

The contravariant transform is a source of the Berezin's contravariant symbol as follows. For a pair $v \in V$, $f \in V^*$, consider a rank-one operator $E_{v,f} : V \to V$ define by the expression $E_{v,f}u = \langle u, f\rangle\, v$. Then, the representation $\hat{\rho}$ (3.13) acts as follows:

$$\hat{\rho}(g_1,g_2)E_{v,f} = E_{v',f'}, \quad \text{where} \quad v' = \rho(g_2)^{-1}v \quad \text{and} \quad f' = \rho^*(g_1)f, \tag{4.13}$$

with the last identity natural meaning $\langle u, f'\rangle = \langle u, \rho^*(g_1)f\rangle = \langle \rho(g_1)u, f\rangle$. Then, the contravariant transform for the vector $w = E_{v,f} \in B(V)$ becomes:

$$\mathcal{M}_{v,f}(a) = \int_X \int_X a(x_1,x_2)\, \hat{\rho}(s(x_1),s(x_2))E_{v,f}\, dx_1\, dx_2. \tag{4.14}$$

Using (4.13), we can directly write the action of operator (4.14) on $u \in V$:

$$\mathcal{M}_{v,f}(a)u = \int_X \int_X a(x_1,x_2)\ \langle \rho(s(x_1))u, f\rangle\ \hat{\rho}(s(x_2)^{-1})v\, dx_1\, dx_2. \tag{4.15}$$

Here we call a the *contravariant symbol* of the operator $\mathcal{M}_{v,f}(a)$.

The contravariant symbol in the sense of Berezin [4–6] appears if V is a Hilbert space with an irreducible square integrable representation ρ. The covariant transform $\mathcal{W}_v$ for an admissible mother wavelet $v \in V$ identifies V with its image $\mathcal{W}_v V$. There is the respective reproducing kernel k_y (3.6) on $\mathcal{W}_v V$. If the representation $\hat{\rho}$ is restricted to the diagonal $\hat{\rho}(g) = \hat{\rho}(g,g)$ of $G \times G$, then the contravariant transform similar to (4.15) is [5, § 1.1]:

$$\begin{aligned}[\mathcal{M}_{v,v}(a)u](y) &= \int_X a(x)\ \langle \rho(s(x))u, v\rangle\ \hat{\rho}(s(x)^{-1})v(y)\, dx\\ &= \int_X a(x)\, u(x)\, k_y(x)\, dx.\end{aligned} \tag{4.16}$$

The last expression is the *Toeplitz operator* $T_a = PaP$ for the projection P on $\mathcal{W}_v$ defined by the integral in the right-hand side of the reproducing formula (3.6).

The explicit formulae connectioning co- and contravariant symbols are known for a long time [5, (1.12); 6, (3.13)]. Within our approach they are consequences of Prop. 4.6, since covariant and contravariant symbols are special cases of covariant and contravariant transforms.

The original Berezin's papers [4–6] (as well as subsequent developments in the context of abstract reproducing kernel spaces) do not assume any group structure. It is possible to obtain important estimations for norm and compactness in this abstract setting. However, the fundamental examples—the Bergman and FSB spaces—considered in those papers are generated by groups, as we have seen above. In particular, the group structure becomes very relevant in the study composition operators generated by an automorphism of the domain [8]. Furthermore, the formula for twisted convolution (4.7) is also based on the underlined group structure and is not possible on a generic reproducing kernel space.

Example 4.9. (i) For Toeplitz operators (3.18) on the Heisenberg group, the contravariant calculus was already investigated in [4] and is still an important tool [8, 9]. The connections between PDO (1.1) and the Toeplitz operators (3.18) was fruitfully exploited in [22, § 4.2] with the following observation: "The Toeplitz operators are to the Bargmann–Fock model as the pseudodifferential operators are to the Schrödinger model". We used such a technique to obtain Calderón–Vaillancourt–type estimations for relative convolutions on exponential nilpotent Lie groups [45].

(ii) For the AHW group $\tilde{G}$, the connection between PDO-type operators (4.3) [50] and Toeplitz-type operators on $L_2(G \times \hat{G})$ shall closely follow the Heisenberg group suit. Yet, no work in this direction is known to me.

(iii) For the Dynin group, I am not aware of any study of contravariant calculus and Toeplitz-type operators. However, it is natural to expect, that their relation to the PDO-like calculus from Example 4.1(iii) and [3, 13–15] shall be similar to the Heisenberg group. However, due to a higher level of non-commutativity it may be not as straightforward as for an AHW group.

(iv) For $SU(1, 1)$ and Toeplitz operators (3.20), the Berezin contravariant symbols was studied in [6], with numerous fruitful developments, cf. [8, 10, 11, 59].

5. Discussion

The moral of the present overview is that there is no a single formula perfectly serving all situations. However, covariant and contravariant transforms provide a general framework which has a rich and flexible inventory. The presented list of different examples prompts further detailed investigation of this approach in various directions.

Appendix A. Covariant Transform: A Road Map

This paper's presentation was illustrated by numerous detailed examples. Here we provide a brief outline of notions and main formulae used in our constructions.

- Induced representations.
  - G is a locally compact group with a right invariant measure dg and the modular function Δ_G.
  - $H \subset G$ is a subgroup with a right invariant measure dh and the modular function Δ_H.

- $X = H\backslash G$ is the homogeneous space of right cosets: $g_1 \sim g_2$ if $g_1 = hg_2$ for $h \in H$.
- $p : G \to H\backslash G$ is the natural projection of an element to its coset.
- $s : H\backslash G \to H$ is a section—a right inverse of p: $p(sx) = x$ for all $x \in H\backslash G$. The map s is not unique and we often can chose it continuous.
- $r : G \to H$ is defined from the identity: $g = r(g)s(p(g))$, $g \in G$.
- $X = H\backslash G$ is a right G-space with the action: $g : x \mapsto x \cdot g = p(s(x) * g)$, $g \in G$, $x \in X$.
- A representation of G induced by a character χ of the subgroup H is $[\rho_\chi(g)f](x) = \chi(r(s(x) * g))\, f(x \cdot g)$.
- An equivalent form of the induced representation is by the right shift $R(g) : F(g') \mapsto F(g'g)$ on a space of functions with the property $F(hg) = \chi(h)F(g)$, for $h \in H$ and $g \in G$.
- Two models of induced representations are connected by lifting $[\mathcal{L}_\chi f](g) = \chi(h)f(p(g))$ and pulling $[\mathcal{P}F](x) = F(s(x))$.

• Covariant transform.
  - For a representation ρ of G in V and an operator $F : V \to U$, the covariant transform is $[\mathcal{W}_F v](g) = F(\rho(g)v)$, where $v \in V$ and $g \in G$.
  - The induced covariant transform is $[\mathcal{W}_F v](x) = F(\rho(s(x))v)$ for $v \in V$, $x \in H\backslash G$.
  - If a mother wavelet f satisfies to the identity $\int_G a(g)\, \rho^*(g)f\, dg = 0$ for some distribution $a(g)$ on G, then any wavelet transforms $\tilde{v} = \langle v, \rho^*(g)f\rangle$ satisfies the identity $D\tilde{v} = 0$, where $D = \int_G \bar{a}(g)\, \Lambda(g)\, dg$ for the left regular representation Λ. This conditions can characterise the image $\mathcal{W}_f V$ among all functions on G or X.
  - Often, the image of (induced) covariant transform has the reproducing property $f(y) = \int_X f(x)\, \bar{k}_y(x)\, dx$, where k_y is the covariant transform of the shifted mother wavelet $\rho(s(y)^{-1})v$.
  - For a representation ρ of G in a vector space V there is a representation $\hat{\rho}(g_1, g_2) : A \mapsto \rho(g_2)^{-1}A\rho(g_1)$ of $G \times G$ on the space $B(V)$ of bounded linear operators on V.
  - The covariant symbol $\tilde{A}(g_1, g_2) = F(\hat{\rho}(g_1, g_2)A) = F(\rho(g_2)^{-1}A\rho(g_1))$ is the covariant transform defined by the representation $\hat{\rho}$ and the operator $F : B(V) \to U$.

• Contravariant transform.
  - For a representation ρ of G and a summable function k on G, the integrated representation is $\rho(k) = \int_G k(g)\, \rho(g)\, dg$.
  - The relative convolution is an integrated representation over a homogeneous space $\rho(k) = \int_X k(x)\, \rho(s(x))\, dx$.
  - The composition of two $\rho(k_1)\rho(k_2)$ relative convolutions produces the twisted convolution $\rho(k_1 \natural k_2) = \rho(k_1)\rho(k_2)$.
  - For an integrated representation or relative convolution ρ and a fixed vector $w \in V$ the contravariant transform of a function k is $\mathcal{M}^\rho_w(k) = \rho(\check{k})w$, where $\check{k}(g) = k(g^{-1})$.
  - The twisted convolution of two wavelet transforms is $\mathcal{W}_{v_1}u_1 \natural \mathcal{W}_{v_2}u_2 = \theta(u_2, v_1)\mathcal{W}_{v_2}u_1$.

References

[1] S. T. Ali, J.-P. Antoine, and J.-P. Gazeau. *Coherent states, wavelets and their generalizations*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 2000. ↑9, 10, 15, 21

[2] P. Aniello. Star products: a group-theoretical point of view. *J. Phys. A* 42 (47):475210, 29, 2009. ↑20

[3] H. Bahouri, C. Fermanian-Kammerer, and I. Gallagher. *Phase-space analysis and pseudodifferential calculus on the Heisenberg group*. Astérisque, vol. 324. SMF, Paris, 2012. ↑2, 3, 18, 23

[4] F. A. Berezin. Wick and anti-Wick symbols of operators. *Mat. Sb. (N.S.)* 86(128):578–610, 1971. Reprinted in [7, pp. 193–227]. ↑11, 13, 16, 22, 23

[5] F. A. Berezin. Covariant and contravariant symbols of operators. *Izv. Akad. Nauk SSSR Ser. Mat.* 36:1134–1167, 1972. Reprinted in [7, pp. 228–261]. ↑12, 16, 22, 23

[6] F. A. Berezin. General concept of quantization. *Comm. Math. Phys.* 40:153–174, 1975. Reprinted in [7, pp. 273–295]. ↑15, 17, 21, 22, 23

[7] F. A. Berezin. *Metod vtorichnogo kvantovaniya*. "Nauka", Moscow, Second, 1986. Edited and with a preface by M. K. Polivanov. ↑25

[8] L. A. Coburn. Berezin transform and Weyl-type unitary operators on the Bergman space. *Proc. Amer. Math. Soc.* 140 (10):3445–3451, 2012. ↑5, 9, 12, 14, 15, 16, 17, 21, 23

[9] L. A. Coburn, J. Isralowitz, and B. Li. Toeplitz operators with BMO symbols on the Segal-Bargmann space. *Trans. Amer. Math. Soc.* 363 (6):3015–3030, 2011. ↑11, 13, 16, 23

[10] L. A. Coburn. Toeplitz operators, quantum mechanics and mean oscillation in the Bergman metric. *Proceedings of Symposia in Pure Mathematics* 51 (1):97–104, 1990. ↑23

[11] L. A. Coburn. Berezin-Toeplitz quantization. *Algebraic mettods in operator theory*, pages 101–108, Birkhäuser Verlag, New York, 1994. ↑23

[12] M. A. de Gosson. *Symplectic methods in harmonic analysis and in mathematical physics*. Pseudo-Differential Operators. Theory and Applications, vol. 7. Birkhäuser/Springer Basel AG, Basel, 2011. ↑2, 10

[13] A. S. Dynin. Pseudodifferential operators on the Heisenberg group. *Dokl. Akad. Nauk SSSR* 225 (6):1245–1248, 1975. ↑1, 3, 4, 5, 8, 14, 18, 23

[14] A. S. Dynin. An algebra of pseudodifferential operators on the Heisenberg groups. Symbolic calculus. *Dokl. Akad. Nauk SSSR* 227 (4):792–795, 1976. ↑1, 3, 4, 5, 14, 16, 18, 23

[15] V. Fischer and M. Ruzhansky. A pseudo-differential calculus on graded Lie groups, Sep. 2012. E-print: `arXiv:1209.2621`. ↑2, 3, 23

[16] G. B. Folland and E. M. Stein. *Hardy spaces on homogeneous group*. Princeton University Press, Princeton, New Jersey, 1982. ↑1, 15

[17] G. B. Folland. *Harmonic analysis in phase space*. Annals of Mathematics Studies, vol. 122. Princeton University Press, Princeton, NJ, 1989. ↑2, 3, 4, 8, 10, 11, 13, 17, 18, 20, 21

[18] G. B. Folland. Meta-Heisenberg groups. In W. O. Bray, P. S. Milojević, and Časlav V. Stanojević (eds.) *Fourier analysis: Analytic and geometric aspects*, pages 121–147, Marcel Dekker, Inc., New York, 1994. ↑1, 3, 5, 8

[19] G. B. Folland. *A course in abstract harmonic analysis*. Studies in Advanced Mathematics. Boca Raton, FL: CRC Press. viii, 276 p. $ 61.95, 1995. ↑5

[20] L. Hörmander. *The analysis of linear partial differential operators* III*: Pseudodifferential operators*. Springer-Verlag, Berlin, 1985. ↑2

[21] R. Howe. On the role of the Heisenberg group in harmonic analysis. *Bull. Amer. Math. Soc. (N.S.)* 3 (2):821–843, 1980. ↑1, 3

[22] R. Howe. Quantum mechanics and partial differential equations. *J. Funct. Anal.* 38 (2):188–254, 1980. ↑1, 3, 4, 13, 17, 20, 23

[23] R. Howe, G. Ratcliff, and N. Wildberger. Symbol mappings for certain nilpotent groups. *Lie group representations, III (College Park, Md., 1982/1983)*, pages 288–320, Springer, Berlin, 1984. ↑1

[24] R. Howe and E.-C. Tan. *Nonabelian harmonic analysis. Applications of* $\mathrm{SL}(2,\mathbf{R})$. Springer-Verlag, New York, 1992. ↑9

[25] A. A. Kirillov. *Elements of the theory of representations*. Springer-Verlag, Berlin, 1976. Translated from the Russian by Edwin Hewitt, Grundlehren der Mathematischen Wissenschaften, Band 220. ↑5, 7, 21

[26] A. A. Kirillov. *Lectures on the orbit method*. Graduate Studies in Mathematics, vol. 64. American Mathematical Society, Providence, RI, 2004. ↑5, 7

[27] A. A. Kirillov and A. D. Gvishiani. *Theorems and problems in functional analysis*. Problem Books in Mathematics. Springer-Verlag, New York, 1982. ↑2, 4

[28] V. V. Kisil. On the algebra of pseudodifferential operators that is generated by convolutions on the Heisenberg group. *Sibirsk. Mat. Zh.* 34 (6):75–85, ii, viii, 1993. (Russian) MR95a:47053. ↑1, 14, 16, 18

[29] V. V. Kisil. Local behavior of two-sided convolution operators with singular kernels on the Heisenberg group. *Mat. Zametki* 56 (2):41–55, 158, 1994. (Russian) MR96a:22015. ↑1, 14

[30] V. V. Kisil. Connection between two-sided and one-sided convolution type operators on a non-commutative group. *Integral Equations Operator Theory* 22 (3):317–332, 1995. MR96d:44004. ↑1

[31] V. V. Kisil. Local algebras of two-sided convolutions on the Heisenberg group. *Mat. Zametki* 59 (3):370–381, 479, 1996. ↑1, 14

[32] V. V. Kisil. Möbius transformations and monogenic functional calculus. *Electron. Res. Announc. Amer. Math. Soc.* 2 (1):26–33, 1996. On-line. ↑9

[33] V. V. Kisil. Analysis in $\mathbf{R}^{1,1}$ or the principal function theory. *Complex Variables Theory Appl.* 40 (2):93–118, 1999. E-print: `arXiv:funct-an/9712003`. ↑5, 12

[34] V. V. Kisil. Relative convolutions. I. Properties and applications. *Adv. Math.* 147 (1):35–73, 1999. E-print: `arXiv:funct-an/9410001`, On-line. Zbl933.43004. ↑1, 4, 17, 19

[35] V. V. Kisil. Wavelets in Banach spaces. *Acta Appl. Math.* 59 (1):79–109, 1999. E-print: `arXiv:math/9807141`, On-line. ↑1, 4, 9, 12, 16

[36] V. V. Kisil. Spectrum as the support of functional calculus. *Functional analysis and its applications*, pages 133–141, Elsevier, Amsterdam, 2004. E-print: `arXiv:math.FA/0208249`. ↑9

[37] V. V. Kisil. Wavelets beyond admissibility. In M. Ruzhansky and J. Wirth (eds.) *Progress in analysis and its applications*, pages 219–225, World Sci. Publ., Hackensack, NJ, 2010. E-print: `arXiv:0911.4701`. ↑4, 9, 21

[38] V. V. Kisil. Covariant transform. *Journal of Physics: Conference Series* 284 (1):012038, 2011. E-print: `arXiv:1011.3947`. ↑4, 21

[39] V. V. Kisil. Erlangen programme at large: an Overview. In S. V. Rogosin and A. A. Koroleva (eds.) *Advances in applied analysis*, pages 1–94, Birkhäuser Verlag, Basel, 2012. E-print: `arXiv:1106.1686`. ↑4, 5, 9, 10, 12, 18, 21

[40] V. V. Kisil. *Geometry of Möbius transformations: Elliptic, parabolic and hyperbolic actions of* $\mathrm{SL}_2(\mathbf{R})$. Imperial College Press, London, 2012. Includes a live DVD. ↑5, 6

[41] V. V. Kisil. Hypercomplex representations of the Heisenberg group and mechanics. *Internat. J. Theoret. Phys.* 51 (3):964–984, 2012. E-print: `arXiv:1005.5057`. ↑8

[42] V. V. Kisil. Induced representations and hypercomplex numbers. *Adv. Appl. Clifford Algebras*:1–24, 2012. E-print: `arXiv:0909.4464`. ↑5

[43] V. V. Kisil. Operator covariant transform and local principle. *J. Phys. A: Math. Theor.* 45:244022, 2012. E-print: `arXiv:1201.1749`. `http://stacks.iop.org/1751-8121/45/244022`. ↑2, 9, 12, 14, 15, 18

[44] V. V. Kisil. The real and complex techniques in harmonic analysis from the covariant transform, 2012. E-print: `arXiv:1209.5072`. ↑4, 10, 15, 21

[45] V. V. Kisil. Boundedness of relative convolutions on nilpotent Lie groups, 2013. E-print: `arXiv:`. ↑23

[46] S. Lang. $\mathrm{SL}_2(\mathbf{R})$. Graduate Texts in Mathematics, vol. 105. Springer-Verlag, New York, 1985. Reprint of the 1975 edition. ↑5, 9, 12, 21

[47] N. K. Nikolski. *Operators, functions, and systems: an easy reading. Vol. 1: Hardy, Hankel, and Toeplitz.* Mathematical Surveys and Monographs, vol. 92. American Mathematical Society, Providence, RI, 2002. Translated from the French by Andreas Hartmann. ↑15, 16

[48] A. Perelomov. *Generalized coherent states and their applications*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1986. ↑9

[49] G. Ratcliff. Symbols and orbits for 3-step nilpotent Lie groups. *J. Funct. Anal.* 62 (1):38–64, 1985. ↑1, 3, 5

[50] M. Ruzhansky and V. Turunen. *Pseudo-differential operators and symmetries. Background analysis and advanced topics*. Pseudo-Differential Operators. Theory and Applications, vol. 2. Birkhäuser Verlag, Basel, 2010. ↑2, 3, 18, 23

[51] M. A. Shubin. *Pseudodifferential operators and spectral theory*. Springer-Verlag, Berlin, Second, 2001. Translated from the 1978 Russian original by Stig I. Andersson. ↑2

[52] I. B. Simonenko. A new general method of investigating linear operator equations of singular integral equation type. I. *Izv. Akad. Nauk SSSR Ser. Mat.* 29:567–586, 1965. ↑14, 16

[53] I. B. Simonenko. A new general method of investigating linear operator equations of singular integral equation type. II. *Izv. Akad. Nauk SSSR Ser. Mat.* 29:757–782, 1965. ↑14

[54] B. Street. An algebra containing the two-sided convolution operators. *Adv. Math.* 219 (1):251–315, 2008. ↑2
[55] M. E. Taylor. *Pseudodifferential operators*. Princeton Mathematical Series, vol. 34. Princeton University Press, Princeton, N.J., 1981. ↑2
[56] M. E. Taylor. *Non commutative microlocal analysis. Part 1*. Mem. of AMS, vol. 313. American Mathematical Society, Providence, R.I., 1984. ↑1
[57] M. E. Taylor. *Noncommutative harmonic analysis*. Mathematical Surveys and Monographs, vol. 22. American Mathematical Society, Providence, RI, 1986. ↑5
[58] A. A. Ungar. *A gyrovector space approach to hyperbolic geometry*. Synthesis Lectures on Mathematics and Statistics, vol. 4. Morgan & Claypool Publishers, Williston, VT, 2009. ↑19
[59] N. L. Vasilevski. *Commutative algebras of Toeplitz operators on the Bergman space*. Operator Theory: Advances and Applications, vol. 185. Birkhäuser Verlag, Basel, 2008. ↑15, 17, 21, 23
[60] H. Weyl. *The theory of groups and quantum mechanics*. Dover, New York, 1950. ↑2
[61] N. J. Wildberger. Weyl quantization and a symbol calculus for abelian groups. *J. Aust. Math. Soc.* 78 (3):323–338, 2005. ↑2, 4, 20

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